AD NUMBER
AD051120
LIMITATION CHANGES
TO: **Approved for public release; distribution is unlimited. Document partially illegible.**
FROM: **Distribution authorized to U.S. Gov't. agencies and their contractors; Administrative/Operational Use; MAR 1950. Other requests shall be referred to Army Research Laboratory, RDRL-LOA-I, Aberdeen Proving Ground, MD 21005. Document partially illegible.**
AUTHORITY
AMRDEC ltr dtd: 22 Jan 2013

original
[redacted]
5-1120

BRL

TECHNICAL NOTE NO. 151

MARCH 1950

ON ESTIMATING BALLISTIC LIMIT AND ITS PRECISION

Abraham Golub
Frank E. Grubbs

BALLISTIC RESEARCH LABORATORIES

ABERDEEN PROVING GROUND, MARYLAND

BALLISTIC RESEARCH LABORATORIES

TECHNICAL NOTE NO. 157

March 1950

ON ESTIMATING BALLISTIC LIMIT AND ITS PRECISION

Project No. TB3-1224F of the Research and Development Division, Ordnance Department

ABERDEEN PROVING GROUND, MARYLAND

TABLE OF CONTENTS

AGolub/FEGrubbs/csr

ON ESTIMATING BALLISTIC LIMIT AND ITS PRECISION

1. Introduction. In current tests of armor plate, AP projectiles are fired at various velocity levels against a given plate until some mixture of complete penetrations and partial penetrations is obtained. The "Ballistic Limit"* of the plate is then estimated somewhat arbitrarily by using the average velocity of two projectiles - one resulting in a complete penetration and the other a partial penetration. It is seen, however, that such a "rule-of-thumb" does not lead to a standard error which may be attached to the estimate of ballistic limit or to an estimate of the standard deviation of the distribution of velocities for the zone of "mixed results". The zone of "mixed results" is an interval extending from the velocity for which the proportion of penetrations would be zero to a velocity at which the proportion of complete penetration would be unity. The purpose of the present note is therefore to investigate a method for estimating the mean or median (50%) velocity, the standard deviation of the distribution of velocities for the zone of mixed results and to approximate standard errors or estimates of precision which can be attached to the above two figures. The general problem here involves the analysis of so-called "sensitivity data".

Problems involving the use of sensitivity data have been dealt with in detail by C. I. Bliss [1], C. West Churchman [2] the Statistical Research Group, Princeton University [3] and others. All these treatments, however, deal with problems for which the levels of stimulus or test can be preassigned or controlled. There is a class of sensitivity problems, however, for which the levels of test can neither be assigned precisely in advance or controlled. A typical problem of this type is the present one of determining the Ballistic Limit velocity of armor plate. For this case, a velocity, for example, of 2000 f/s may be aimed at (by adjustment of the weight of propellant powder), but due to a random distribution of velocities for a fixed charge, the velocity actually attained may be, say, 2015 f/s. Thus, it is not feasible to control velocity such that, for example, 5 projectiles could be fired at a velocity of 2000 f/s,

*Ballistic Limit has, in many instances, been defined somewhat loosely as "that velocity at which a given type of projectile will penetrate a given (thickness and type of) armor plate". However, it turns out that for a series of AP projectiles fired at an appropriate constant velocity part of the projectiles will completely penetrate the plate and the remainder will not, the proportion penetrating depending on the level of velocity. Thus, for problems of this type it becomes necessary to regard the "Ballistic Limit" as a parameter of the probability distribution involving the proportion of successes for various levels of velocity.

another 5 rounds at 2025 f/s, etc. Once an AP projectile has been fired and its velocity measured, however, then it can be said that the velocity attained is known rather precisely and is thus "free of error". In the execution of a test for determining the Ballistic Limit of a given armor plate, therefore, the results of such a test usually yield a set of distinct projectile velocities or testing levels. Assuming that each projectile has a "critical velocity" at or above which the projectile will penetrate the plate and below which it will fail to penetrate, and also that these critical velocities follow the Normal Distribution Law, then the general problem may be to determine for any given velocity the probability of a penetration. However, it is usually sufficient in many practical cases to determine that velocity for which the probability of penetration is one-half (defined hereinafter as the Ballistic Limit velocity). The remainder of this note is concerned with how the mean or Ballistic Limit velocity and standard deviation of the normally distributed critical velocities may be estimated from a unique set of velocities, and how the precision of our estimates may be approximated. Although our primary interest centers around tests of armor, it is apparent that the methods studied herein are generally applicable to other problems of the type considered in this note.

2. Determination of the Likelihood Function. As a result of test we have an observed set of generally distinct velocities, v_i, and a statement for each velocity that the projectile penetrated or failed to penetrate the plate. If the true probabilities of penetration at these velocities are p_i, the probability of the observed set of observations may be written in the form

(1) $P = \prod_i p_i^{\delta_i} q_i^{(1-\delta_i)}$ $\qquad \delta_i = 0$ or 1 for the different levels, v_i, of test.

where under the assumption of the Normal Distribution of critical velocities

$$p_i = \int_{-\infty}^{t_i} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt = 1 - q_i = \text{the chance of a complete}$$

penetration at the velocity, v_i,

$$t_i = \frac{v_i - \mu}{\sigma} = \text{the deviation in velocity from } \mu \text{ expressed}$$

in standard units and μ and σ are the unknown mean or 50% velocity and standard deviation in velocity, we desired to estimate.

The quantity δ_i is a discrete random variable which takes the value 1 for a complete penetration and the value 0 for a partial or incomplete penetration. (At the velocity v_i, the quantity δ_i has an expected or mean value of p_i).

To estimate the parameters μ and σ, employing R. A. Fisher's Method of Maximum Likelihood, it is customary to maximize the Likelihood function L = Log P i.e.

$$(2) \quad L = \sum_i \left\{ \delta_i \log p_i + (1-\delta_i) \log q_i \right\}$$

3. Method of Maximum Likelihood for the Estimation of μ and σ. In order to maximize L we equate to zero the partial derivatives of L with respect to μ and σ, and then solve for the estimates $\hat{\mu}$ and $\hat{\sigma}$. [The caret is used to denote an estimate of a population parameter].

To maximize the function L, the derivatives of p_i and q_i with respect to μ and σ will be needed. They are:

$$\frac{\partial p_i}{\partial \mu} = -\frac{Z_i}{\sigma} \qquad \frac{\partial q_i}{\partial \mu} = \frac{Z_i}{\sigma}$$

$$\frac{\partial p_i}{\partial \sigma} = -\frac{t_i Z_i}{\sigma} \qquad \frac{\partial q_i}{\partial \sigma} = \frac{t_i Z_i}{\sigma}$$

where $Z_i = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$ where $t_i = \frac{v_i - \mu}{\sigma}$

The maximum likelihood equations are then readily found to be

$$(3) \quad \frac{\partial L}{\partial \mu} = \frac{1}{\sigma} \sum_i \left\{ (1-\delta_i) \frac{Z_i}{q_i} - \delta_i \frac{Z_i}{p_i} \right\} = 0$$

$$(4) \quad \frac{\partial L}{\partial \sigma} = \frac{1}{\sigma} \sum_i \left\{ (1-\delta_i) \frac{t_i Z_i}{q_i} - \delta_i \frac{t_i Z_i}{p_i} \right\} = 0$$

For an observed set of unique velocities, however, we substitute the actual δ_i (i.e. 0 or 1) for each velocity and thus (1), (2), (3) and (4) may conveniently be rearranged and rewritten for computational purposes as

(1a) $P = p_1 p_2 \cdots p_n \; q_1, q_2, \cdots q_m$ where

n= number of penetrations and m = number of non-penetrations.

(2a) $L = \sum_{i=1}^{n} \log p_i + \sum_{j=1}^{m} \log q_j$ (i for complete penetrations, j for failures to penetrate)

(3a) $$\frac{\partial L}{\partial \mu} = \frac{1}{\sigma}\left[\sum_j \frac{Z_j}{q_j} - \sum_i \frac{Z_i}{p_i}\right] = 0$$

(4a) $$\frac{\partial L}{\partial \sigma} = \frac{1}{\sigma}\left[\sum_j \frac{t_j Z_j}{q_j} - \sum_i \frac{t_i Z_i}{p_i}\right] = 0$$

In order to solve the above set of equations [(3a) and (4a)] for μ and σ the standard iteration method for estimating two parameters from the maximum likelihood principle will be employed. This procedure is particularly effective provided close first estimates of μ and σ can be found. In this connection, if a suitable number of rounds have been fired then the use of probability paper is expeditious in obtaining first estimates, by grouping the velocities, plotting the observed proportion of complete penetrations as a function of the observed velocities, and estimating μ and σ from the line graphed. Should the number of rounds fired be few, say five, a graphical first estimate may be obtained by plotting equation (3a) for varying σ and equation (4a) for varying μ. Intersection of these two curves will give a solution or a good first estimate to use. It is conceivable that for a given set of observations, however, practical solutions of equations (3a) and (4a) may not exist. That is to say, we will not be concerned with values of the standard deviation, σ, for which $\sigma < 0$.

The appropriate simultaneous equations for iteration on μ and σ are

(5) $$-\frac{\partial L}{\partial \mu_0} = \frac{\partial^2 L}{\partial \mu_0^2} \Delta\mu + \frac{\partial^2 L}{\partial \mu_0 \partial \sigma_0} \Delta\sigma$$

(6) $$-\frac{\partial L}{\partial \sigma_0} = \frac{\partial^2 L}{\partial \mu_0 \partial \sigma_0} \Delta\mu + \frac{\partial^2 L}{\partial \sigma_0^2} \Delta\sigma$$

Where $\Delta\mu = \hat{\mu} - \mu_0$ and $\Delta\sigma = \hat{\sigma} - \sigma_0$ give the approximate errors in μ_0 and σ_0 or the increments to be added to the estimates (μ_0, σ_0) in the iteration process.

It is evident from (5) and (6) that the second partial derivatives of L are needed for the iteration and they are:

$$(7)\quad -\frac{\partial^2 L}{\partial \mu^2} = \frac{1}{\sigma^2}\left[\sum_j \frac{t_j z_j}{q_j} - \sum_j \frac{z_j^2}{q_j^2} - \sum_i \frac{t_i z_i}{p_i} - \sum_i \frac{z_i^2}{p_i^2}\right]$$

$$(8)\quad \frac{\partial^2 L}{\partial \mu \partial \sigma} = \frac{1}{\sigma^2}\left[\left[\sum_j \frac{t_j^2 z_j}{q_j} - \sum_j \frac{t_j z_j^2}{q_j^2} - \sum_i \frac{t_i^2 z_i}{p_i} - \sum_i \frac{t_i z_i^2}{p_i^2}\right] - \sum_j \frac{z_j}{q_j} + \sum_i \frac{z_i}{p_i}\right]$$

$$(9)\quad \frac{\partial^2 L}{\partial \sigma^2} = \frac{1}{\sigma^2}\left[\sum_j \frac{t_j^3 z_j}{q_j} - \sum_j \frac{t_j z_j}{q_j} - \sum_j \frac{t_j^2 z_j^2}{q_j^2} - \sum_i \frac{t_i^3 z_i}{p_i} + \sum_i \frac{t_i z_i}{p_i} - \sum_i \frac{t_i^2 z_i^2}{p_i^2}\right] + \frac{1}{\sigma^2}\left[\sum_i \frac{t_i z_i}{p_i} - \sum_j \frac{t_j z_j}{q_j}\right]$$

By examination of these equations it is apparent (see numerical example below) that the computational work involved can be accomplished somewhat readily on a desk calculator, since all expressions involved are functions of the following columns of factors.

	t_i	t_i^2	t_i^3	z_i/p_i	z_i/q_i	z_i^2/p_i^2	z_j^2/q_j^2
q_j's							
p_i's							

Values of t_i, Z_i/p_i and Z_i/q_i are given in Tables I and II and were taken from reference [4].

4. The Approximate Variances of $\hat{\mu}$ and $\hat{\sigma}$. Approximations for the variances of $\hat{\mu}$ and $\hat{\sigma}$ may be obtained, in accordance with existing Maximum Likelihood theory, from the expected values of the second partial derivatives of the likelihood L as expressed in (2).

The second partial derivatives of the likelihood L as expressed in (2) are:

(10) $$\frac{\partial^2 L}{\partial \mu^2} = \frac{1}{\sigma^2} \sum_i \left\{ (1-\delta_i) \frac{t_i Z_i}{q_i} - (1-\delta_i) \frac{Z_i^2}{q_i^2} - \delta_i \frac{t_i Z_i}{p_i} - \delta_i \frac{Z_i^2}{p_i^2} \right\}$$

(11) $$\frac{\partial^2 L}{\partial \mu \partial \sigma} = \frac{1}{\sigma^2} \sum_i \left\{ (1-\delta_i) \frac{t_i^2 Z_i}{q_i} - (1-\delta_i) \frac{t_i Z_i^2}{q_i^2} - \delta_i \frac{t_i^2 Z_i}{p_i} - \delta_i \frac{t_i Z_i^2}{p_i^2} - (1-\delta_i)\frac{z_i}{q_i} + \delta_i \frac{z_i}{p_i} \right\}$$

(12) $$\frac{\partial^2 L}{\partial \sigma^2} = \frac{1}{\sigma^2} \sum_i \left\{ (1-\delta_i) \frac{t_i^3 Z_i}{q_i} - (1-\delta_i) \frac{t_i Z_i}{q_i} - (1-\delta_i) \frac{t_i^2 Z_i^2}{q_i^2} - \delta_i \frac{t_i^3 Z_i}{p_i} + \delta_i \frac{t_i Z_i}{p_i} - \delta_i \frac{t_i^2 Z_i^2}{p_i^2} \right\} - (1-\delta_i)\frac{t_i z_i}{q_i} + \delta_i \frac{t_i z_i}{p_i}$$

Since the expected value of δ_i [i.e. $E(\delta_i)$] is p_i, we obtain from (3), (4), (10), (11), and (12) the following:

(13) $$E\left(\frac{\partial L}{\partial \mu}\right) = \frac{1}{\sigma} \sum_i \left\{ Z_i - Z_i \right\} = 0$$

(14) $$E\left(\frac{\partial L}{\partial \sigma}\right) = \frac{1}{\sigma} \sum_i \left\{ t_i Z_i - t_i Z_i \right\} = 0$$

(15) $$E\left(\frac{\partial^2 L}{\partial \mu^2}\right) = \frac{1}{\sigma^2} \sum_i \left\{ -\frac{Z_i^2}{q_i} - \frac{Z_i^2}{p_i} \right\}$$

(16) $$E\left(\frac{\partial^2 L}{\partial \mu \partial \sigma}\right) = \frac{1}{\sigma^2} \sum_i \left\{ -\frac{t_i Z_i^2}{q_i} - \frac{t_i Z_i^2}{p_i} \right\}$$

$$(17) \quad E\left(\frac{\partial^2 L}{\partial \sigma^2}\right) = \frac{1}{\sigma^2} \sum_i \left\{ -\frac{w_i^2 z_i^2}{q_i} - \frac{t_i^2 z_i^2}{p_i} \right\}$$

Assuming that approximately

$$-E\left(\frac{\partial^2 L}{\partial \mu^2}\right) = E\left[\left(\frac{\partial L}{\partial \mu}\right)^2\right]$$

$$-E\left(\frac{\partial^2 L}{\partial \mu \partial \sigma}\right) = E\left[\left(\frac{\partial L}{\partial \mu}\right)\left(\frac{\partial L}{\partial \sigma}\right)\right]$$

$$-E\left(\frac{\partial^2 L}{\partial \sigma^2}\right) = E\left[\left(\frac{\partial L}{\partial \sigma}\right)^2\right]$$

S.S. Wilks

and writing

$$\Lambda_{\mu\mu} = -E\left(\frac{\partial^2 L}{\partial \mu^2}\right)$$

$$\Lambda_{\mu\sigma} = -E\left(\frac{\partial^2 L}{\partial \mu \partial \sigma}\right)$$

$$\Lambda_{\sigma\sigma} = -E\left(\frac{\partial^2 L}{\partial \sigma^2}\right)$$

(which are the approximate variances and covariance of $\frac{\partial L}{\partial \mu}$ and $\frac{\partial L}{\partial \sigma}$) then from a theorem on Maximum Likelihood estimates the variance-covariance matrix of $\hat{\mu}$ and $\hat{\sigma}$ may be obtained as

$$\begin{Vmatrix} \Lambda_{\mu\mu} & \Lambda_{\mu\sigma} \\ \Lambda_{\mu\sigma} & \Lambda_{\sigma\sigma} \end{Vmatrix}^{-1} = \begin{Vmatrix} \Lambda^{\mu\mu} & \Lambda^{\mu\sigma} \\ \Lambda^{\mu\sigma} & \Lambda^{\sigma\sigma} \end{Vmatrix}$$

in which $\Lambda^{\mu\mu} \cong \sigma_{\hat{\mu}}^2$, the variance of $\hat{\mu}$; $\Lambda^{\sigma\sigma} \cong \sigma_{\hat{\sigma}}^2$, the variance of $\hat{\sigma}$; and $\Lambda^{\mu\sigma}$ is the approximate covariance of $\hat{\mu}$ and $\hat{\sigma}$.

5. Example. In firing five rounds of a given projectile at a given plate the following observations were recorded.

Velocity	Condition of Impact
2433	Non-Penetration
2415	Non-Penetration
2415	Non-Penetration
2453	Penetration
2423	Penetration

Using the graphical method of obtaining a first estimate, as discussed in paragraph 3, we obtain for plausible first estimate, $\mu_0 = 2440$ and $\sigma_0 = 20$. We are now in a position to employ the iteration method of solution and set up a systematic tabulation as follows:

	N.P.	P	t_i	t_i^2	t_i^3	Z_j/q_j	Z_i/p_i	Z_j^2/q_j^2	Z_i^2/p_i^2
q_j	2433		-.350	.1225	-.0429	.589		.3469	
	2415		-1.250	1.5625	-1.9531	.204		.0416	
	2415		-1.250	1.5625	-1.9531	.204		.0416	
p_i		2453	.650	.4225	.2746		.435		.1892
		2423	-.850	.7225	-.6141		1.406		1.9768

$$\sigma\left(\frac{\partial L}{\partial \mu}\right)_0 = \sum_j \frac{Z_j}{q_j} - \sum_i \frac{Z_i}{p_i} = -.8440$$

$$\sigma\left(\frac{\partial L}{\partial \sigma}\right)_0 = \sum_j \frac{t_j Z_j}{q_j} - \sum_i \frac{t_i Z_i}{p_i} = .1962$$

$$\sigma\left(\frac{\partial^2 L}{\partial \mu^2}\right)_0 = \frac{1}{\sigma}\left[\sum_j \frac{t_j Z_j}{q_j} - \sum_j \frac{Z_j^2}{q_j^2} - \sum_i \frac{t_i Z_i}{p_i} - \sum_i \frac{Z_i^2}{p_i^2}\right] \text{+ additional terms (see eq (9), page 7)} = -.1200$$

$$\sigma\left(\frac{\partial^2 L}{\partial \mu \partial \sigma}\right)_0 = \frac{1}{\sigma}\left[\sum_j \frac{t_j^2 Z_j}{q_j} - \sum_j \frac{t_j Z_j^2}{q_j^2} - \sum_i \frac{t_i^2 Z_i}{p_i} - \sum_i \frac{t_i Z_i^2}{p_i^2}\right] = .0646$$

$$\sigma\left(\frac{\partial^2 L}{\partial \sigma^2}\right)_0 = \frac{1}{\sigma}\left[\sum_j \frac{t_j^3 Z_j}{q_j} - \sum_j \frac{t_j Z_j}{q_j} - \sum_j \frac{t_j^2 Z_j^2}{q_j^2} - \sum_i \frac{t_i^3 Z_i}{p_i} + \sum_i \frac{t_i Z_i}{p_i} - \sum_i \frac{t_i^2 Z_i^2}{p_i^2}\right] \text{+ additional terms (see eq (9), page 7)} = -.0978$$

Equations (5) and (6) now become

$$.8440 = -.1200\,\Delta\mu + .0646\,\Delta\sigma$$
$$-.1962 = .0646\,\Delta\mu - .0978\,\Delta\sigma$$

Solving for $\Delta\mu$ and $\Delta\sigma$ we get

$$\Delta\mu = -9.2;\ \Delta\sigma = -4.1$$

Our second estimate of μ and σ thus become approximately $\hat{\mu}_1 = 2430$ and $\hat{\sigma}_1 = 16$. Repeating the above process twice more, we finally obtain estimates to the nearest ft/sec.; namely $\hat{\mu} = 2431$, $\hat{\sigma} = 15$.

To determine the approximate variances of $\hat{\mu}$ and $\hat{\sigma}$ we compute

$$E\left(\frac{\partial^2 L}{\partial \mu^2}\right) = \frac{1}{\sigma^2}\sum_i \left\{-\frac{Z_i^2}{q_i} - \frac{Z_i^2}{p_i}\right\} = \frac{1}{225}\,[-2.301] = -.01023$$

$$E\left(\frac{\partial^2 L}{\partial \mu \partial \sigma}\right) = \frac{1}{\sigma^2}\sum_i \left\{-\frac{t_i Z_i^2}{q_i} - \frac{t_i Z_i^2}{p_i}\right\} = \frac{1}{225}\,[.7310] = .00325$$

$$E\left(\frac{\partial^2 L}{\partial \sigma^2}\right) = \frac{1}{\sigma^2}\sum_i \left\{-\frac{t_i^2 Z_i^2}{q_i} - \frac{t_i^2 Z_i^2}{p_i}\right\} = \frac{1}{225}\,[-1.7659] = -.00785$$

Thus $\Lambda_{\mu\mu} = .01023$

$\Lambda_{\mu\sigma} = -.00325$

$\Lambda_{\sigma\sigma} = .00785$

The variance-covariance matrix is given by

$$\begin{Vmatrix} .01023 & -.00325 \\ -.00325 & .00785 \end{Vmatrix}^{-1} = \begin{Vmatrix} 112.6 & 46.6 \\ 46.6 & 146.7 \end{Vmatrix}$$

yielding

$$\sigma^2_{\hat{\mu}} = 112.6 \qquad \sigma^2_{\hat{\sigma}} = 146.7$$

$$\sigma_{\hat{\mu}} = 10.6 \qquad \sigma_{\hat{\sigma}} = 12.1$$

6. Conclusion. For the particular class of problems employing sensitivity data with which we are concerned in this report it is possible to determine by the Method of Maximum Likelihood estimates of the parameters μ and σ of the assumed underlying Normal Distribution. Moreover, it is possible to determine the approximate variances of these estimates.

It is recommended, however, that in the execution of the test care be taken to insure a good "zone of mixed results", i.e. a range of velocities which yield an entanglement of penetrations and non-penetrations. Probably the best way to obtain a good "zone of mixed results" would be the use of an "Up and Down" Method [3] of firing.

References:

[1] C. I. Bliss, Ann. Appl. Biol., XXII (1935) p. 134.
[2] C. W. Churchman, Ann. of Math. Stat., Vol. XV, No. 1 1944, pp 90-96.
[3] App. Math. Panel, Report No. 101-1R, 1944.
[4] W. J. Dixon and A. M. Mood, Jour. of Amer. Stat. Assoc. Vol. 43, 1948, p. 109.

TABLE I

VALUES OF Z /p

t	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.798	0.792	0.785	0.779	0.773	0.766	0.760	0.754	0.748	0.741
0.1	0.735	0.729	0.723	0.717	0.711	0.705	0.699	0.693	0.687	0.681
0.2	0.675	0.669	0.663	0.657	0.652	0.646	0.640	0.634	0.629	0.623
0.3	0.617	0.612	0.606	0.600	0.595	0.589	0.584	0.578	0.573	0.567
0.4	0.562	0.556	0.551	0.546	0.540	0.535	0.530	0.525	0.519	0.514
0.5	0.509	0.504	0.499	0.494	0.489	0.484	0.479	0.474	0.469	0.464
0.6	0.459	0.454	0.449	0.445	0.440	0.435	0.430	0.426	0.421	0.417
0.7	0.412	0.407	0.403	0.398	0.394	0.389	0.385	0.381	0.376	0.372
0.8	0.368	0.363	0.359	0.355	0.351	0.346	0.342	0.338	0.334	0.330
0.9	0.326	0.322	0.318	0.314	0.310	0.306	0.303	0.299	0.295	0.291
1.0	0.288	0.284	0.280	0.277	0.273	0.269	0.266	0.262	0.259	0.255
1.1	0.252	0.249	0.245	0.242	0.239	0.235	0.232	0.229	0.226	0.223
1.2	0.219	0.216	0.213	0.210	0.207	0.204	0.201	0.198	0.195	0.193
1.3	0.190	0.187	0.184	0.181	0.179	0.176	0.173	0.171	0.168	0.165
1.4	0.163	0.160	0.158	0.155	0.153	0.150	0.148	0.146	0.143	0.141
1.5	0.139	0.137	0.134	0.132	0.130	0.128	0.126	0.124	0.121	0.119
1.6	0.117	0.115	0.113	0.111	0.110	0.108	0.106	0.104	0.102	0.100
1.7	0.098	0.097	0.095	0.093	0.092	0.090	0.088	0.087	0.085	0.083
1.8	0.082	0.080	0.079	0.077	0.076	0.074	0.073	0.072	0.070	0.069
1.9	0.068	0.066	0.065	0.064	0.062	0.061	0.060	0.059	0.058	0.056
2.0	0.055	0.054	0.053	0.052	0.051	0.050	0.049	0.048	0.047	0.046
2.1	0.045	0.044	0.043	0.042	0.041	0.040	0.039	0.038	0.038	0.037
2.2	0.036	0.035	0.034	0.034	0.033	0.032	0.031	0.031	0.030	0.029
2.3	0.029	0.028	0.027	0.027	0.026	0.025	0.025	0.024	0.024	0.023
2.4	0.023	0.022	0.022	0.021	0.020	0.020	0.019	0.019	0.019	0.018
2.5	0.018	0.017	0.017	0.016	0.016	0.016	0.015	0.015	0.014	0.014
2.6	0.014	0.013	0.013	0.013	0.012	0.012	0.012	0.011	0.011	0.011
2.7	0.010	0.010	0.010	0.010	0.009	0.009	0.009	0.009	0.008	0.008
2.8	0.008	0.008	0.008	0.007	0.007	0.007	0.007	0.007	0.006	0.006
2.9	0.006	0.006	0.006	0.005	0.005	0.005	0.005	0.005	0.005	0.005
3.0	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.003	0.003
3.1	0.003	0.003	0.003	0.003	0.003	0.003	0.003	0.003	0.003	0.002
3.2	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002
3.3	0.002	0.002	0.002	0.002	0.002	0.001	0.001	0.001	0.001	0.001
3.4	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001
3.5	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001
3.6	0.001	0.001	0.001	0.001	0.001	0.001	0.000			

For negative values of t, p and q are interchanged, that is

$$\frac{Z(-t)}{p(-t)} = \frac{Z(t)}{q(t)} \quad \text{and} \quad \frac{Z(-t)}{q(-t)} = \frac{Z(t)}{p(t)}$$

TABLE II

VALUES OF Z /q

t	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.798	0.804	0.811	0.817	0.824	0.830	0.836	0.843	0.849	0.856
0.1	0.863	0.869	0.876	0.882	0.889	0.896	0.902	0.909	0.916	0.923
0.2	0.929	0.936	0.943	0.950	0.957	0.964	0.970	0.977	0.984	0.991
0.3	0.998	1.005	1.012	1.019	1.026	1.033	1.040	1.047	1.054	1.062
0.4	1.069	1.076	1.083	1.090	1.097	1.105	1.112	1.119	1.126	1.134
0.5	1.141	1.148	1.156	1.163	1.171	1.178	1.185	1.193	1.200	1.207
0.6	1.215	1.222	1.230	1.237	1.245	1.253	1.260	1.268	1.275	1.283
0.7	1.290	1.298	1.306	1.313	1.321	1.329	1.336	1.344	1.352	1.360
0.8	1.367	1.375	1.383	1.391	1.399	1.406	1.414	1.422	1.430	1.438
0.9	1.446	1.454	1.461	1.469	1.477	1.485	1.493	1.501	1.509	1.517
1.0	1.525	1.533	1.541	1.549	1.557	1.565	1.573	1.581	1.590	1.598
1.1	1.606	1.614	1.622	1.630	1.638	1.646	1.655	1.663	1.671	1.679
1.2	1.687	1.696	1.704	1.712	1.720	1.729	1.737	1.745	1.754	1.762
1.3	1.770	1.779	1.787	1.795	1.804	1.812	1.820	1.829	1.838	1.846
1.4	1.854	1.862	1.871	1.879	1.888	1.896	1.905	1.913	1.922	1.930
1.5	1.938	1.947	1.955	1.964	1.972	1.981	1.990	1.998	2.007	2.015
1.6	2.024	2.033	2.041	2.050	2.058	2.067	2.076	2.084	2.093	2.102
1.7	2.110	2.119	2.128	2.136	2.145	2.154	2.162	2.171	2.180	2.188
1.8	2.197	2.206	2.215	2.223	2.232	2.241	2.250	2.258	2.267	2.276
1.9	2.285	2.294	2.303	2.311	2.320	2.329	2.338	2.346	2.355	2.364
2.0	2.373	2.381	2.390	2.399	2.408	2.417	2.426	2.435	2.444	2.453
2.1	2.462	2.470	2.479	2.488	2.497	2.506	2.515	2.524	2.533	2.542
2.2	2.551	2.560	2.569	2.578	2.587	2.596	2.605	2.614	2.623	2.632
2.3	2.641	2.650	2.659	2.668	2.677	2.687	2.696	2.705	2.714	2.723
2.4	2.732	2.741	2.750	2.759	2.768	2.777	2.786	2.795	2.805	2.814
2.5	2.823	2.832	2.841	2.850	2.859	2.868	2.878	2.887	2.896	2.905
2.6	2.914	2.923	2.932	2.942	2.951	2.960	2.969	2.978	2.987	2.997
2.7	3.006	3.015	3.024	3.033	3.043	3.052	3.061	3.070	3.079	3.089
2.8	3.098	3.107	3.116	3.126	3.135	3.144	3.153	3.163	3.172	3.181
2.9	3.190	3.200	3.209	3.218	3.227	3.237	3.246	3.255	3.265	3.274
3.0	3.283	3.292	3.302	3.311	3.320	3.330	3.339	3.348	3.358	3.367
3.1	3.376	3.386	3.395	3.404	3.413	3.423	3.432	3.441	3.451	3.460
3.2	3.470	3.479	3.488	3.498	3.507	3.516	3.526	3.535	3.544	3.554
3.3	3.563	3.573	3.582	3.591	3.601	3.610	3.620	3.629	3.638	3.648
3.4	3.657	3.667	3.676	3.685	3.695	3.704	3.714	3.723	3.732	3.742
3.5	3.751	3.761	3.770	3.780	3.789	3.799	3.808	3.817	3.827	3.836
3.6	3.846	3.855	3.865	3.874	3.884	3.893	3.902	3.912	3.992	3.931
3.7	3.940	3.950	3.959	3.969	3.978	3.988	3.997	4.007	4.016	4.026
3.8	4.035	4.045	4.054	4.064	4.073	4.083	4.092	4.102	4.111	4.121
3.9	4.130	4.140	4.149	4.159	4.169	4.178	4.188	4.197	4.206	4.216
4.0	4.226	4.235	4.245	4.254	4.264	4.273	4.283	4.292	4.302	4.312

For negative values t, p and q are interchanged, that is

$$\frac{Z(-t)}{p(-t)} = \frac{Z(t)}{q(t)} \quad \text{and} \quad \frac{Z(-t)}{q(-t)} = \frac{Z(t)}{p(t)}$$

DISTRIBUTION LIST

No. of Copies	
4	Chief of Ordnance Washington 25, D. C. ATTN: ORDTB-Bal Sec
2	Commanding Officer Watertown Arsenal Watertown, Mass.
2	Commanding Officer Frankford Arsenal Philadelphia, Pa.
2	Commanding Officer Picatinny Arsenal Dover, N. J.
2	Commanding Officer Detroit Arsenal Detroit, Mich.
2	Chief, Bureau of Ordnance Navy Department Washington 25, D. C. ATTN: Re3
2	Commanding Officer Naval Proving Ground Dahlgren, Va.

Infosec

From: No-Reply@amrdec.army.mil
Sent: Tuesday, January 22, 2013 10:00 AM
To: Infosec
Subject: AMRDEC Safe Access File Exchange Delivery Notice

Tracy Landfried, tracy.landfried@us.army.mil has granted you access to a file(s).

File Description: AD0051120 aka BRL-TN-151 See 2 pages behind title page for email traffic on current release status.

Changed from CUI to U2.

Is this sufficient to change the status of document?

Note this is also a much better copy than DTIC has.

Tracy Landfried
DSN: 298-6840 (

-----Original Message-----
From: Burton, Lawrence W CIV (US)
Sent: Friday, January 18, 2013 1:18 PM
To: Goode, Jennifer L CIV (US)
Subject: FW: Request for foreign release. (UNCLASSIFIED)

Jen,

Per the attached email trail, both Matt Burkins and I concur that the info in the report is public releasable.

Larry

-----Original Message-----
From: Goode, Jennifer L CIV (US)
Sent: Thursday, January 17, 2013 8:49 AM
To: Burton, Lawrence W CIV (US)
Subject: FW: Request for foreign release. (UNCLASSIFIED)

Larry,
I need your expertise.

Thanks,
Jen

Jennifer L. Goode
Security Specialist
Army Research Laboratory

-----Original Message-----
From: Letendre, Louise A CIV (US)
Sent: Tuesday, January 15, 2013 12:45 PM
To: Bailey, Donald R CIV (US)
Subject: FW: Request for foreign release. (UNCLASSIFIED)

Mr. Bailey: The attached BRL Technical Note, published in 1950 and designated at the time as releasable to US Gov't and contractors, has been Our researcher has requested assistance in determining if the document is still under limited distribution as it is over 60 years old. The document is identified in our catalog but is not available except by application to the library staff, nor is it in DTIC. Request you initiate a review the document for purposes of regarding. A copy of the catalog entry and the original request follow below:

Catalog entry: On Estimation Ballistic Limit and Its Precision
Golub, A; Grubbs, FE
Accession Number AD0051120
Corporate Author ARMY BALLISTIC RESEARCH LAB ABERDEEN PROVING GROUND MD

Author(s) Golub, A, Grubbs, FE
Unclassified Title On Estimation Ballistic Limit and Its Precision
Report Date 19500300
Report Number(s) BRL-TN-151
Pagination 14
Descriptive Note Technical note no. 151
Supplementary Note This report is available in electronic format to authorized personnel. Please submit a request to the library for a digital copy.
Dist./Avail. Stmt. U.S. GOVT. AND THEIR CONTRACTORS, AVAILABILITY: DOCUMENT PARTIALLY ILLEGIBLE
Source Code 050750